Resit Exam — Complex Analysis

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Duration: 3 hours

Instructions

1. The test consists of 6 questions; answer all of them.
2. Each question gets 15 points and the number of points for each subquestion is indicated at the beginning of the subquestion. 10 points are "free" and the total number of points is divided by 10. The final grade will be between 1 and 10.
3. The use of books, notes, and calculators is not allowed.

Question 1 (15 points)

Consider the function $f(z) = ze^{-iz}$ with z in $\mathbb{C}$.

a. (7 points) Write $f(z)$ as a sum of a real and an imaginary part, in other words, in the form $u(x,y) + iv(x,y)$ where $z = x + iy$.

b. (8 points) Use the Cauchy-Riemann equations to show that $f(z)$ is entire.

Question 2 (15 points)

Consider the function

$$f(z) = \frac{1}{z(z+1)}.$$

a. (9 points) Find the Laurent series for $f(z)$ in $0 < |z| < 1$.

b. (6 points) What is the type of the singularity of $f(z)$ at 0? Explain your answer.

Question 3 (15 points)

Consider the function

$$f(z) = \frac{e^{iz}}{z^2+4}.$$

a. (6 points) Compute the residue of $f(z)$ at each one of the singularities of the function.

b. (9 points) Evaluate

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx.$$

Question 4 (15 points)

a. (7 points) Given the function

$$f(z) = z\,(z+2)\left(z - \frac{i}{2}\right)^2$$

compute the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the positively oriented circular contour with $|z| = 1$.

b. (8 points) Use Rouché's theorem to show that the polynomial $P(z) = z^3 - \frac{1}{2}z^2 + 1$ has exactly 3 roots in the disk $|z| < 2$.

Question 5 (15 points)

We denote by $\operatorname{Log} z$ the principal value of the logarithm $\log z$.

a. (7 points) Prove that $\operatorname{Log}(e^z) = z$ *if and only if* $-\pi < \operatorname{Im} z \leq \pi$.

b. (8 points) Construct a branch of $\log(z+4)$ that is analytic at the point $z = -5$ and takes the value $7\pi i$ there.

Question 6 (15 points)

The generalized Cauchy integral formula gives that if $f(z)$ is analytic inside and on a circle C_r of radius r centered at z_0 then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\,dz.$$

a. (9 points) Prove that if $f(z)$ is analytic *inside and on* a circle C_r of radius r centered at z_0 and if $|f(z)| \leq M$ for all z on C_r, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

b. (6 points) Prove that if $f(z)$ is analytic for all z in the domain $|z - z_0| < R$ and if $|f(z)| \leq M$ in the same domain, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

[Hint: Apply the result of the previous subquestion. Nevertheless, be careful that you cannot directly apply this result with $r = R$ since $f(z)$ is not assumed analytic on the circle C_R with $|z - z_0| = R$.]

End of the test.